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Global Geometric Deformations of the Virasoro Algebra, Current and Affine Algebras by Krichever–Novikov Type Algebras

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Abstract In two earlier articles we constructed algebraic-geometric families of genus one (i.e. elliptic) Lie algebras of Krichever–Novikov type. The considered algebras are vector fields, current and affine Lie algebras. These families deform the Witt algebra, the Virasoro algebra, the classical current, and the affine Kac–Moody Lie algebras respectively. The constructed families are not equivalent (not even locally) to the trivial families, despite the fact that the classical algebras are formally rigid. This effect is due to the fact that the algebras are infinite dimensional. In this article the results are reviewed and developed further. The constructions are induced by the geometric process of degenerating the elliptic curves to singular cubics. The algebras are of relevance in the global operator approach to the Wess–Zumino–Witten–Novikov models appearing in the quantization of Conformal Field Theory.

Keywords Deformations of algebras · Rigidity · Wess–Zumino–Witten–Novikov models · Krichever–Novikov algebras · Conformal field theory

1 Introduction

Deformation theory plays a crucial role in all branches of mathematics and physics. In physics the mathematical theory of deformations has proved to be a powerful tool in modeling physical reality. The concepts *symmetry* and *deformations* are considered to be two fundamental guiding principle for developing the physical theory further. From the mathematical point of view considering deformed objects will give additional information about the original object itself, in particular, how is its relation to “neighbouring” objects. This

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can be made precise with the notion of moduli space, classifying inequivalent objects of the same type. The moduli space should be equipped with a "geometric" structure such that "nearby points" should be also "nearby" in the sense of deforming the structure of the initial object. Moreover, assuming that such a moduli space exists, its dimension should be equal to the number of inequivalent deformation directions. Maybe there exists even a deformation family containing every possible deformation.

Clearly, this general remarks are rather vague. To make them more precise first one has to be more precise about the structure to deform. A very famous and well-developed domain is the deformation theory of complex analytic structures of a compact complex manifold M. We do not have the place to recall here this theory, but refer only to [17] for results and details. Let us only mention that a fundamental role is played by the first cohomology space $\mathrm{H}^1(M, T_M)$ of M with values in the holomorphic tangent sheaf T_M. In particular, if this space is trivial, M will be rigid, i.e. it cannot be deformed in something which is not isomorphic to it.

Here we will deal with deformations of Lie algebras, in particular of such of infinite dimension. Formal deformations of arbitrary rings and associative algebras, and the related cohomology questions, were first investigated by Gerstenhaber, in a series of articles [8–10]. The notion of deformation was applied to Lie algebras by Nijenhuis and Richardson [15, 16].

The cohomology space related to deformations of a Lie algebra $\mathcal{L}$ is the Lie algebra two-cohomology $\mathrm{H}^2(\mathcal{L},\mathcal{L})$ of $\mathcal{L}$ with values in the adjoint module. We will explain this in Sect. 3. As long as the Lie algebra is finite-dimensional, the relation is rather tight. In particular, if the cohomology space vanishes, the Lie algebra will be rigid in all respects.

But the algebras which are e.g. of relevance in Conformal Field Theory, integrable systems related to partial differential equations, etc. are typically infinite dimensional. We are interested here in these algebras. We showed in two articles [6, 7] that the relation to cohomology is not so tight anymore. In particular we constructed nontrivial geometric deformation families for the Witt algebra (resp. its universal central extension the Virasoro algebra) and for the current algebras (resp. their central extensions the affine algebras), despite the fact that the cohomology spaces for those algebras are trivial and hence the algebras are formally rigid [4, 14]. This is a phenomena which in finite dimension cannot occur.

Here we report on our results and the constructions to be found in [6, 7] and continue our investigation. The Witt algebra is the algebra consisting of those meromorphic vector fields on the Riemann sphere which are holomorphic outside $\{0,\infty\}$. A basis and the associated structure is given by

$$l_n = z^{n+1}\frac{d}{dz},\quad n\in\mathbb{Z},\quad \text{with Lie bracket } [l_n,l_m]=(m-n)\,l_{n+m}.$$

The Virasoro algebra is its universal central extension

$$[l_n,l_m]=(m-n)l_{n+m}+\frac{1}{12}(m^3-m)\delta_{n,-m}t,\qquad [l_n,t]=0,$$

with t an additional basis element which is central.

Furthermore we consider the case of current algebras $\bar{\mathfrak{g}}=\mathfrak{g}\otimes\mathbb{C}[z^{-1},z]$ and their central extensions $\hat{\mathfrak{g}}$, the *affine Lie algebras*. Here $\mathfrak{g}$ is a finite-dimensional Lie algebra (which for simplicity we assume to be simple). With the Cartan–Killing form β the central extension $\hat{\mathfrak{g}}$ is the vector space $\bar{\mathfrak{g}}\oplus t\,\mathbb{C}$ endowed with the Lie bracket

$$[x\otimes z^n, y\otimes z^m]=[x,y]\otimes z^{n+m}-\beta(x,y)\cdot n\cdot\delta_m^{-n}\cdot t,$$
$$[t,\hat{\mathfrak{g}}]=0,\qquad x,y\in\mathfrak{g},\ n,m\in\mathbb{Z}.$$

As already mentioned, these algebras are rigid.

The families we construct, appear as families of higher-genus multi-point algebras of Krichever–Novikov type, see Sect. 4 for their definitions. Hence, they are not just abstract families, but families obtained by geometric processes. The results obtained do not have only relevance in deformation theory of algebras, but they also are of importance in areas where vector fields, current, and affine algebras play a role.

A very prominent application is two-dimensional conformal field theory (CFT) and its quantization. It is well-known that the Witt algebra, the Virasoro algebra, the current algebras, the affine algebras, and their representations are of fundamental importance for CFT on the Riemann sphere (i.e. for genus zero), see [1]. Krichever and Novikov [11–13] proposed in the case of higher genus Riemann surfaces (with two insertion points) the use of global operator fields which are given with the help of the Lie algebra of vector fields of Krichever–Novikov type, certain related algebras, and their representations (see Sect. 4 below).

Their approach was extended by Schlichenmaier to the multi-point situation (i.e. an arbitrary number of insertion points was allowed) [18–21]. The necessary central extensions where constructed. Higher genus multi-point current and affine algebras were introduced [23]. These algebras consist of meromorphic objects on a Riemann surface which are holomorphic outside a finite set A of points. The set A is divided into two disjoint subsets I and O. With respect to some possible interpretation of the Riemann surface as the world-sheet of a string, the points in I are called *in-points*, the points in O are called *out-points*, corresponding to incoming and outgoing free strings. The world-sheet itself corresponds to possible interaction. This splitting introduces an almost-graded structure (see Sect. 4) for the algebras and their representations. Such an almost-graded structure is needed to construct representations of relevance in the context of the quantization of CFT, e.g. highest weight representations, fermionic Fock space representations, etc.

In the following we give more information on a special model. In the process of quantization of conformal fields one has to consider families of algebras and representations over the moduli space of compact Riemann surfaces (or equivalently, of smooth projective curves over $\mathbb{C}$) of genus g with N marked points. Models of most importance in CFT are the Wess–Zumino–Witten–Novikov models (WZWN). Tsuchiya, Ueno and Yamada [29] gave a sheaf version of WZWN models over the moduli space. In [27, 28] Schlichenmaier and Sheinman developed a global operator version. In this context of particular interest is the situation $I = \{P_1, \ldots, P_K\}$, the marked points we want to vary, and $O = \{P_\infty\}$, a reference point. We obtain families of algebras over the moduli space $\mathcal{M}_{g,K+1}$ of curves of genus g with $K+1$ marked points, and we are exactly in the middle of the main subject of this article. In [27] and [28] it is shown that there exists a global operator description of WZWN models with the help of the Krichever Novikov objects at least over a dense open subset of the moduli space. Starting from families of representations $\mathcal{V}$ of families of higher genus affine algebras (see Sect. 4 for their definition) the vector bundle of conformal blocks can be defined as the vector bundle with fibre (over the moduli point $b = [(M, \{P_1, \ldots, P_K\}, \{P_\infty\})]$) the quotient space of the fibre $\mathcal{V}_b$ modulo the subspace generated by the vectors obtained by the action of those elements of the affine algebra which vanish at the reference point P_∞ (i.e. the fibre is the space of coinvariants of this subalgebra).

The bundle of conformal blocks carries a connection called the *Knizhnik–Zamolodchikov connection*. In its definition an important role is played by the Sugawara construction which associates to representations of affine algebras representations of the (almost-graded) centrally extended vector field algebras, see [26]. A certain subspace of the vector field algebra (assigned to the moduli point b) corresponds to tangent directions on the moduli space $\mathcal{M}_{g,K+1}$ at the point b.

Now clearly, the following question is of fundamental importance. What happens if we approach the boundary of the moduli space? The boundary components correspond to curves with singularities. Resolving the singularities yields curves of lower genera. By geometric degeneration we obtain families of (Lie) algebras containing a lower genus algebra (or sometimes a subalgebra of it), corresponding to a suitable collection of marked points, as special element. Or reverting the perspective, we obtain a typical situation of the deformation of an algebra corresponding in some way to a lower genus situation, containing higher genus algebras as the other elements in the family. Such kind of geometric degenerations are of fundamental importance if one wants to prove Verlinde type formula via factorization and normalization technique, see [29].

By a maximal degeneration a collection of $\mathbb{P}^1(\mathbb{C})$'s will appear. Indeed, the examples considered in this article are exactly of this type. The deformations appear as families of vector fields and current algebras which are naturally defined over the moduli space of genus one curves (i.e. of elliptic curves, or equivalently of complex one-dimensional tori) with two marked points. These deformations are associated to geometric degenerations of elliptic curves to singular cubic curves. The desingularization (or normalization) of their singularities will yield the projective line as normalization. We will end up with algebras related to the genus zero case. The full geometric picture behind the degeneration was discussed in [22]. In particular, we like to point out, that even if one starts with two marked points, by passing to the boundary of the moduli space one is forced to consider more points (now for a curve of lower genus).

2 Deformations of Lie Algebras

In the physics literature a Lie algebra $\mathcal{L}$ is often given in terms of generators and structure constants. Let V be a finite- or infinite dimensional complex vector space with basis $\{T_a\}_{a\in J}$ then a Lie algebra structure on V can be given by the structure equations, i.e. the Lie bracket,

$$[T_a, T_b] = {\sum_{c\in J}}' C^c_{a,b} T_c, \quad a, b \in J, \tag{2.1}$$

with structure constants $C^c_{a,b} \in \mathbb{C}$. The symbol $\sum'$ denotes that for fixed $a, b \in J$ only for finitely many c the coefficient $C^c_{a,b} \neq 0$. In terms of structure constants the necessary and sufficient conditions for $\mathcal{L}$ being a Lie algebra (i.e. the anti-symmetry and the Jacobi identity) can be written as

$$\begin{gathered} C^c_{a,b} + C^c_{b,a} = 0, \quad a, b, c \in J, \\ \sum_{l\in J} (C^l_{a,b} C^d_{l,c} + C^l_{b,c} C^d_{l,a} + C^l_{c,a} C^d_{l,b}) = 0, \quad a, b, c, d \in J. \end{gathered} \tag{2.2}$$

Deforming the Lie algebra structure corresponds intuitively to making the system of coefficients $\{C^c_{a,b}\}$ depending on one or more parameters.

In a more compact manner a Lie algebra $\mathcal{L}$, i.e. its bracket $[.,.]$, might be written with an anti-symmetric bilinear form

$$\mu_0 : \mathcal{L} \times \mathcal{L} \to \mathcal{L}, \qquad \mu_0(x, y) = [x, y],$$

fulfilling certain additional conditions corresponding to the Jacobi identity. Consider on the same vector space $\mathcal{L}$ is modeled on, a family of Lie structures

$$\mu_t = \mu_0 + t \cdot \phi_1 + t^2 \cdot \phi_2 + \cdots , \tag{2.3}$$

with bilinear maps $\phi_i : \mathcal{L} \times \mathcal{L} \to \mathcal{L}$ such that $\mathcal{L}_t := (\mathcal{L}, \mu_t)$ is a Lie algebra and $\mathcal{L}_0$ is the Lie algebra we started with. The family $\{\mathcal{L}_t\}$ is a *deformation* of $\mathcal{L}_0$.

Up to this point we did not specify the "parameter" t. Indeed, different choices are possible.

(1) The parameter t might be a variable which allows to plug in numbers $\alpha \in \mathbb{C}$. In this case $\mathcal{L}_\alpha$ is a Lie algebra for every α for which the expression (2.3) is defined. The family can be considered as deformation over the affine line $\mathbb{C}[t]$ or over the convergent power series $\mathbb{C}\{\{t\}\}$. The deformation is called a *geometric* or an *analytic deformation* respectively.
(2) We consider t as a formal variable and we allow infinitely many terms in (2.3). It might be the case that μ_t does not exist if we plug in for t any other value different from 0. In this way we obtain deformations over the ring of formal power series $\mathbb{C}[[t]]$. The corresponding deformation is a *formal deformation*.
(3) The parameter t is considered as an infinitesimal variable, i.e. we take $t^2 = 0$. We obtain *infinitesimal deformations* defined over the quotient $\mathbb{C}[X]/(X^2) = \mathbb{C}[[X]]/(X^2)$.

We could even consider more general situations for the parameter space. See Appendix 1 for a general mathematical definition of a deformation.

There is always the trivially deformed family given by $\mu_t = \mu_0$ for all values of t. Two families μ_t and μ'_t deforming the same μ_0 are *equivalent* if there exists a linear automorphism (with the same vagueness about the meaning of t)

$$\psi_t = id + t \cdot \alpha_1 + t^2 \cdot \alpha_2 + \cdots \tag{2.4}$$

with $\alpha_i : \mathcal{L} \to \mathcal{L}$ linear maps such that

$$\mu'_t(x, y) = \psi_t^{-1}(\mu_t(\psi_t(x), \psi_t(y))). \tag{2.5}$$

A Lie algebra $(\mathcal{L}, \mu_0)$ is called *rigid* if every deformation μ_t of μ_0 is locally equivalent to the trivial family. Intuitively, this says that $\mathcal{L}$ cannot be deformed.

The word "locally" in the definition of rigidity means that we only consider the situation for t "near 0". Of course, this depends on the category we consider. As on the formal and the infinitesimal level there exists only one closed point, i.e. the point 0 itself, every deformation over $\mathbb{C}[[t]]$ or $\mathbb{C}[X]/(X^2)$ is already local. This is different on the geometric and analytic level. Here it means that there exists an open neighborhood U of 0 such that the family restricted to it is equivalent to the trivial one. In particular, this implies $\mathcal{L}_\alpha \cong \mathcal{L}_0$ for all $\alpha \in U$.

Clearly, a question of fundamental interest is to decide whether a given Lie algebra is rigid. Moreover, the question of rigidity will depend on the category we consider. Depending on the set-up we will have to consider infinitesimal, formal, geometric, and analytic rigidity. If the algebra is not rigid, one would like to know whether there exists a moduli space of (inequivalent) deformations. If so, what is its structure, dimension, etc.?

As explained in the introduction, deformation problems and moduli space problems are related to adapted cohomology theories. To a certain extend (in particular for the finite-dimensional case) this is also true for deformations of Lie algebras. But as far as geometric and algebraic deformations are concerned it is wrong for infinite dimensional Lie algebras as our examples show.

The corresponding relations to cohomology will be explained in Sect. 3. To see later why the results are different in the infinite dimensional case, let us first discuss the finite-dimensional case. Let $\mathcal{L}$ be a finite-dimensional Lie algebra of dimension n over $\mathbb{C}$ and denote the underlying vector space by V. The structure constants from (2.1) $\{C_{a,b}^{c}\}_{a,b,c=1,\ldots,n}$ are elements of $\mathbb{C}^{n^3}$. The conditions (2.2), which are necessary and sufficient that (2.1) defines a Lie algebra, are algebraic equations. The vanishing set $Lalg_n$ (i.e. the set of common zeros of these equations) in $\mathbb{C}^{n^3}$ "parameterizes" the possible Lie algebra structures on the n-dimensional vector space V. As the conditions are algebraic the vanishing set will be a (not necessarily irreducible) variety. In fact it would be better to talk about $Lalg_n$ as a scheme, as one should better consider the not necessarily reduced structure on $Lalg_n$.

The Lie structure μ is a bilinear map $V \times V \to V$ and the structure constants might be considered as elements of $V^* \otimes V^* \otimes V$ with V^* the dual space of V.

If we make a change of basis, the structure constants will change. The two set of structure constants will define isomorphic Lie algebras. The corresponding effect can be described by a linear automorphism $\Phi \in \mathrm{Gl}(V)$. It will define an action on $V^* \otimes V^* \otimes V$ by

$$(\Phi \star \mu)(x, y) = \Phi(\mu(\Phi^{-1}(x), \Phi^{-1}(y))). \tag{2.6}$$

If μ corresponds to a Lie algebra structure, $\Phi \star \mu$ will also be a Lie algebra. Hence $\Phi\star$ will be an action on $Lalg_n$

The Lie algebras (V, μ) and (V, μ') are isomorphic iff μ and μ' are in the same orbit under this $\mathrm{Gl}(V)$ action. On the level of structure constants, i.e. after fixing a basis in V, we obtain a $\mathrm{Gl}(n)$ action on $Lalg_n$. In this way the isomorphy classes of Lie algebras of dimension n correspond exactly to the $\mathrm{Gl}(n)$ orbits of $Lalg_n$.

The variety $Lalg_n$ decomposes into different orbits under the $\mathrm{Gl}(n)$-action. Let x_0 be a point in $Lalg_n$ (defining the Lie structure μ_0). All "nearby" Lie structures μ correspond to points x near x_0. Of course the $\mathrm{Gl}(n)$ orbit of x_0 passes through x_0. If all points in an open neighborhood of x_0 lie in this orbit then this implies that all "nearby" Lie structures μ_t are isomorphic to μ_0. A Lie algebra is called rigid in the orbit sense, if the corresponding orbit is Zariski open in $Lalg_n$.[1] In particular, rigidity in the orbit sense implies rigidity in the geometric and analytic sense.

Intuitively the "moduli space" of finite-dimensional Lie structures should correspond to the orbit space under the $\mathrm{Gl}(n)$-action. But as in the boundary of certain orbits there might be different orbits this will need some modification. Indeed, the problem of the geometric structure of the "moduli space" is rather delicate and as we will not need it here, we will not discuss it, see Bjar and Laudal [2].

Back to our Lie algebras of arbitrary dimensions. Special types of deformations are *jump deformations*. They are typically given as families over a parameter space (parameterized e.g. by t) around 0, such that $\mathcal{L}_t \cong \mathcal{L}_{t'}$ as long as $t, t' \neq 0$, but $\mathcal{L}_t \ncong \mathcal{L}_0$. In the finite-dimensional case, considered above, the element $\mathcal{L}_0$ will be necessarily a boundary point of the orbit of $\mathcal{L}_t$, $t \neq 0$ which does not lie in the orbit itself. This says it is an element in the Zariski closure of the orbit but not of the orbit itself. Sometimes in physics one talks about *contractions*. This language is dual to the language of jump deformations. Here $\mathcal{L}_0$ is

[1] A subset is called Zariski open if it is the complement of the vanishing set of finitely many algebraic equations. Zariski open subset are always open in the usual topology.

a contraction of the isomorphy type of $\mathcal{L}_t$, $t \neq 0$. Moreover, in the finite-dimensional case the possible contractions of $\mathcal{L}$ are given by the boundary points of its $\mathrm{Gl}(n)$ orbit.

3 Cohomological Description

For Lie algebra deformations the relevant cohomology space is $\mathrm{H}^2(\mathcal{L}, \mathcal{L})$, the space of Lie algebra two-cohomology classes with values in the adjoint module $\mathcal{L}$.

Recall that these cohomology classes are classes of two-cocycles modulo coboundaries. An antisymmetric bilinear map $\phi : \mathcal{L} \times \mathcal{L} \to \mathcal{L}$ is a Lie algebra *two-cocycle* if $d_2\phi = 0$, or expressed explicitly

$$\phi([x, y], z) + \phi([y, z], x) + \phi([z, x], y) - [x, \phi(y, z)] + [y, \phi(z, x)] - [z, \phi(x, y)] = 0. \tag{3.1}$$

The map ϕ will be a *coboundary* if there exists a linear map $\psi : \mathcal{L} \to \mathcal{L}$ with

$$\phi(x, y) = (d_1\psi)(x, y) := \psi([x, y]) - [x, \psi(y)] + [y, \psi(x)]. \tag{3.2}$$

If we write the Jacobi identity for μ_t given by (2.3) then it can be immediately verified that the first non-vanishing ϕ_i has to be a two-cocycle in the above sense. Furthermore, if μ_t and μ'_t are equivalent then the corresponding ϕ_i and ϕ'_i are cohomologous, i.e. their difference is a coboundary.

The following results are well-known:

(1) $\mathrm{H}^2(\mathcal{L}, \mathcal{L})$ classifies infinitesimal deformations of $\mathcal{L}$ [8–10].
(2) If $\dim \mathrm{H}^2(\mathcal{L}, \mathcal{L}) < \infty$ then all formal deformations of $\mathcal{L}$ up to equivalence can be realized in this vector space [5].
(3) If $\mathrm{H}^2(\mathcal{L}, \mathcal{L}) = 0$ then $\mathcal{L}$ is infinitesimally and formally rigid (this follows directly from (1) and (2)).
(4) If $\dim \mathcal{L} < \infty$ then $\mathrm{H}^2(\mathcal{L}, \mathcal{L}) = 0$ implies that $\mathcal{L}$ is also rigid in the geometric and analytic sense [8–10, 15, 16].

As our examples show, without the condition $\dim \mathcal{L} < \infty$ point 3 is not true anymore.

For the Witt algebra $\mathcal{W}$ one has $\mathrm{H}^2(\mathcal{W}, \mathcal{W}) = 0$ ([4], see also [6]). Hence it is formally rigid. For the classical current algebras $\bar{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}[z^{-1}, z]$ with $\mathfrak{g}$ a finite-dimensional simple Lie algebra, Lecomte and Roger [14] showed that $\bar{\mathfrak{g}}$ is formally rigid. Nevertheless, for both types of algebras, including their central extensions, we obtained deformations which are both locally geometrically and analytically non-trivial [6, 7]. Hence they are not rigid in the geometric and analytic sense. These families will be described in the following.

4 Krichever–Novikov Algebras

Our geometric families will be families of algebras of Krichever–Novikov type. These algebras play an important role in a global operator approach to higher genus Conformal Field Theory.

They are generalizations of the Virasoro algebra, the current algebras and all their related algebras. Let M be a compact Riemann surface of genus g, or in terms of algebraic geometry, a smooth projective curve over $\mathbb{C}$. Let $N, K \in \mathbb{N}$ with $N \geq 2$ and $1 \leq K < N$. Fix

$$I = (P_1, \ldots, P_K), \quad \text{and} \quad O = (Q_1, \ldots, Q_{N-K})$$

disjoint ordered tuples of distinct points ("marked points", "punctures") on the curve. In particular, we assume $P_i \neq Q_j$ for every pair (i, j). The points in I are called the *in-points*, the points in O the *out-points*. Sometimes we consider I and O simply as sets and set $A = I \cup O$ as a set.

Here we will need the following algebras. Let $\mathcal{A}$ be the associative algebra consisting of those meromorphic functions on M which are holomorphic outside the set of points A with point-wise multiplication. Let $\mathcal{L}$ be the Lie algebra consisting of those meromorphic vector fields which are holomorphic outside of A with the usual Lie bracket of vector fields. The algebra $\mathcal{L}$ is called the *vector field algebra of Krichever–Novikov type*. In the two point case they were introduced by Krichever and Novikov [11–13]. The corresponding generalization to the multi-point case was done in [18–21]. Obviously, both $\mathcal{A}$ and $\mathcal{L}$ are infinite dimensional algebras.

Furthermore, we will need the *higher-genus multi-point current algebra of Krichever–Novikov type*. We start with a complex finite-dimensional Lie algebra $\mathfrak{g}$ and endow the tensor product $\overline{\mathcal{G}} = \mathfrak{g} \otimes_{\mathbb{C}} \mathcal{A}$ with the Lie bracket

$$[x \otimes f, y \otimes g] = [x, y] \otimes f \cdot g, \quad x, y \in \mathfrak{g}, \ f, g \in \mathcal{A}. \tag{4.1}$$

The algebra $\overline{\mathcal{G}}$ is the higher genus current algebra. It is an infinite dimensional Lie algebra and might be considered as the Lie algebra of $\mathfrak{g}$-valued meromorphic functions on the Riemann surface with only poles outside of A.

The classical genus zero and $N = 2$ point case is given by the geometric data

$$M = \mathbb{P}^1(\mathbb{C}) = S^2, \qquad I = \{z = 0\}, \qquad O = \{z = \infty\}. \tag{4.2}$$

In this case the algebras are the well-known algebras of Conformal Field Theory (CFT). For the function algebra we obtain $\mathcal{A} = \mathbb{C}[z^{-1}, z]$, the algebra of Laurent polynomials. The vector field algebra $\mathcal{L}$ is the Witt algebra $\mathcal{W}$ generated by

$$l_n = z^{n+1}\frac{d}{dz}, \quad n \in \mathbb{Z}, \quad \text{with Lie bracket } [l_n, l_m] = (m - n)\, l_{n+m}. \tag{4.3}$$

The current algebra $\overline{\mathcal{G}}$ is the standard current algebra $\bar{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}[z^{-1}, z]$ with Lie bracket

$$[x \otimes z^n, y \otimes z^m] = [x, y] \otimes z^{n+m}, \quad x, y \in \mathfrak{g}, \ n, m \in \mathbb{Z}. \tag{4.4}$$

In the classical situation the algebras are obviously graded by taking as degree $\deg l_n := n$ and $\deg x \otimes z^n := n$. For higher genus there is usually no grading. But it was observed by Krichever and Novikov in the two-point case that a weaker concept, an almost-graded structure, will be enough to develop an interesting theory of representations (Verma modules, etc.). Let $\mathcal{A}$ be an (associative or Lie) algebra admitting a direct decomposition as vector space $\mathcal{A} = \bigoplus_{n\in\mathbb{Z}} \mathcal{A}_n$. The algebra $\mathcal{A}$ is called an *almost-graded* algebra if (1) $\dim \mathcal{A}_n < \infty$ and (2) there are constants R and S such that

$$\mathcal{A}_n \cdot \mathcal{A}_m \subseteq \bigoplus_{h=n+m+R}^{n+m+S} \mathcal{A}_h, \quad \forall n, m \in \mathbb{Z}. \tag{4.5}$$

The elements of $\mathcal{A}_n$ are called *homogeneous elements of degree n*. By exhibiting a special basis, for the multi-point situation such an almost grading was introduced in [18–21]. Essentially, this is done by fixing the order of the basis elements at the points in I in a certain manner and in O in a complementary way to make them unique up to scaling. In the following we will give an explicit description of the basis elements for those genus zero and one situation we need. Hence, we will not recall their general definition but only refer to the above quoted articles.

Proposition 4.1 ([18, 21]) *The algebras $\mathcal{L}$, $\mathcal{A}$, and $\overline{\mathcal{G}}$ are almost-graded. The almost-grading depends on the splitting $A = I \cup O$.*

In the construction of infinite dimensional representations of these algebras with certain desired properties (generated by a vacuum, irreducibility, unitarity, etc.) one is typically forced to "regularize" a "naive" action to make it well-defined. Examples of importance in CFT are the fermionic Fock space representations which are constructed by taking semi-infinite forms of a fixed weight.

From the mathematical point of view, with the help of a prescribed procedure one modifies the action to make it well-defined. On the other hand, one has to accept that the modified action in compensation will only be a projective Lie action. Such projective actions are honest Lie actions for suitable centrally extended algebras. In the classical case they are well-known. The unique non-trivial (up to equivalence and rescaling) central extension of the Witt algebra $\mathcal{W}$ is the Virasoro algebra $\mathcal{V}$:

$$[l_n, l_m] = (m-n)l_{n+m} + \frac{1}{12}(m^3 - m)\delta_{n,-m}\, t, \qquad [l_n, t] = 0. \tag{4.6}$$

Here t is an additional element of the central extension which commutes with all other elements. For the current algebra $\mathfrak{g} \otimes \mathbb{C}[z^{-1}, z]$ for $\mathfrak{g}$ a simple Lie algebra with Cartan–Killing form β, it is the corresponding affine Lie algebra $\hat{\mathfrak{g}}$ (or, untwisted affine Kac–Moody algebra):

$$\begin{aligned} &[x \otimes z^n, y \otimes z^m] = [x, y] \otimes z^{n+m} - \beta(x, y) \cdot n \cdot \delta_m^{-n} \cdot t, \\ &[t, \hat{\mathfrak{g}}] = 0, \quad x, y \in \mathfrak{g},\ n, m \in \mathbb{Z}. \end{aligned} \tag{4.7}$$

The additional terms in front of the elements t are 2-cocycles of the Lie algebras with values in the trivial module $\mathbb{C}$. Indeed, for a Lie algebra V central extensions are classified (up to equivalence) by the second Lie algebra cohomology $\mathrm{H}^2(V, \mathbb{C})$ of V with values in the trivial module $\mathbb{C}$. Similar to the above, the bilinear form $\psi : V \times V \to \mathbb{C}$ is called Lie algebra 2-cocycle iff ψ is antisymmetric and fulfills the cocycle condition

$$0 = d_2\psi(x, y, z) := \psi([x, y], z) + \psi([y, z], x) + \psi([z, x], y). \tag{4.8}$$

It will be a coboundary if there exists a linear form $\kappa : V \to \mathbb{C}$ such that

$$\psi(x, y) = (d_1\kappa)(x, y) := \kappa([x, y]). \tag{4.9}$$

To extend the classical cocycles to the Krichever–Novikov type algebras they first have to be given in geometric terms. Geometric versions of the 2-cocycles are given as follows (see [24] and [25] for details). For the vector field algebra $\mathcal{L}$ we take

$$\gamma_{S,R}(e, f) := \frac{1}{24\pi \mathrm{i}} \int_{C_S} \left(\frac{1}{2}(e''' f - e f''') - R \cdot (e' f - e f') \right) dz. \tag{4.10}$$

Here the integration path C_S is a loop separating the in-points from the out-points and R is a holomorphic projective connection (see [6, Def. 4.2]) to make the integrand well-defined. For the current algebra $\overline{\mathcal{G}}$ we take

$$\gamma_S(x \otimes f, y \otimes g) = \beta(x, y)\frac{1}{2\pi \mathrm{i}}\int_{C_S} f dg. \tag{4.11}$$

The reader should be warned. For the classical algebras, i.e. the Witt and the current algebras for the simple Lie algebras $\mathfrak{g}$, there exists up to rescaling and equivalence only one non-trivial central extension. This will be the Virasoro algebra for the Witt algebra and the affine Kac–Moody algebra for the current algebra respectively. This is not true anymore for higher genus or/and the multi-point situation. But it was shown in [24] and [25] that (again up to equivalence and rescaling) there exists only one non-trivial central extension which allows to extend the almost-grading by giving the element t a degree in such a way that it will also be almost-graded. This unique extension will be given by the geometric cocycles (4.10), (4.11).

5 The Geometric Families

5.1 Complex Torus

Let $\tau \in \mathbb{C}$ with $\Im\tau > 0$ and L be the lattice

$$L = \langle 1, \tau\rangle_{\mathbb{Z}} := \{m + n \cdot \tau \mid m, n \in \mathbb{Z}\} \subset \mathbb{C}. \tag{5.1}$$

The complex one-dimensional torus is the quotient $T = \mathbb{C}/L$. It carries a natural structure of a complex manifold coming from the structure of $\mathbb{C}$. It will be a compact Riemann surface of genus 1.

The field of meromorphic functions on T is generated by the doubly-periodic Weierstraß $\wp$ function and its derivative $\wp'$ fulfilling the differential equation

$$(\wp')^2 = 4(\wp - e_1)(\wp - e_2)(\wp - e_3) = 4\wp^3 - g_2\wp - g_3, \tag{5.2}$$

with

$$\Delta := {g_2}^3 - 27{g_3}^2 = 16(e_1 - e_2)^2(e_1 - e_3)^2(e_2 - e_3)^2 \neq 0. \tag{5.3}$$

Furthermore,

$$g_2 = -4(e_1e_2 + e_1e_3 + e_2e_3), \qquad g_3 = 4(e_1e_2e_3). \tag{5.4}$$

The numbers e_i are pairwise distinct, can be given as

$$\wp\left(\frac{1}{2}\right) = e_1, \qquad \wp\left(\frac{\tau}{2}\right) = e_2, \qquad \wp\left(\frac{\tau+1}{2}\right) = e_3, \tag{5.5}$$

and fulfill

$$e_1 + e_2 + e_3 = 0. \tag{5.6}$$

The function $\wp$ is an even meromorphic function with poles of order two at the points of the lattice and holomorphic elsewhere. The function $\wp'$ is an odd meromorphic function with poles of order three at the points of the lattice and holomorphic elsewhere. $\wp'$ has zeros of order one at the points $1/2$, $\tau/2$ and $(1+\tau)/2$ and all their translates under the lattice.

We have to pass here to the algebraic-geometric picture. The map

$$T \to \mathbb{P}^2(\mathbb{C}), \qquad z \bmod L \mapsto \begin{cases} (\wp(z) : \wp'(z) : 1), & z \notin L, \\ (0:1:0), & z \in L \end{cases} \tag{5.7}$$

realizes T as a complex-algebraic smooth curve in the projective plane. As its genus is one it is an elliptic curve. The affine coordinates are $X = \wp(z,\tau)$ and $Y = \wp'(z,\tau)$. From (5.2) it follows that the affine part of the curve can be given by the smooth cubic curve defined by

$$Y^2 = 4(X-e_1)(X-e_2)(X-e_3) = 4X^3 - g_2 X - g_3 =: f(X). \tag{5.8}$$

The point at infinity on the curve is the point $\infty = (0:1:0)$.

We consider the algebras of Krichever–Novikov type corresponding to the elliptic curve and possible poles at $\bar{z} = \bar{0}$ and $\bar{z} = \overline{1/2}$ [2] (respectively in the algebraic-geometric picture, at the points ∞ and $(e_1, 0)$).

5.2 Vector Field Algebra

First we consider the vector field algebra $\mathcal{L}$. A basis of the vector field algebra is given by

$$V_{2k+1} := (X-e_1)^k Y \frac{d}{dX}, \qquad V_{2k} := \frac{1}{2} f(X)(X-e_1)^{k-2} \frac{d}{dX}, \quad k \in \mathbb{Z}. \tag{5.9}$$

If we vary the points e_1 and e_2 (and accordingly $e_3 = -(e_1+e_2)$) we obtain families of curves and associated families of vector field algebras. At least this is the case as long as the curves are non-singular. To describe the families in detail consider the following straight lines

$$D_s := \{(e_1, e_2) \in \mathbb{C}^2 \mid e_2 = s \cdot e_1\}, \quad s \in \mathbb{C}, \qquad D_\infty := \{(0, e_2) \in \mathbb{C}^2\}, \tag{5.10}$$

and the open subset

$$B = \mathbb{C}^2 \setminus (D_1 \cup D_{-1/2} \cup D_{-2}) \subset \mathbb{C}^2. \tag{5.11}$$

The curves are non-singular exactly over the points of B. Over the exceptional D_s at least two of the e_i are the same. For the vector field algebra we obtain

$$[V_n, V_m] = \begin{cases} (m-n)V_{n+m}, & n, m \text{ odd}, \\ (m-n)\big(V_{n+m} + 3e_1 V_{n+m-2} \\ \quad + (e_1-e_2)(e_1-e_3)V_{n+m-4}\big), & n, m \text{ even}, \\ (m-n)V_{n+m} + (m-n-1)3e_1 V_{n+m-2} \\ \quad + (m-n-2)(e_1-e_2)(e_1-e_3)V_{n+m-4}, & n \text{ odd}, \ m \text{ even}. \end{cases} \tag{5.12}$$

In fact these relations define Lie algebras for every pair $(e_1, e_2) \in \mathbb{C}^2$. We denote by $\mathcal{L}^{(e_1,e_2)}$ the Lie algebra corresponding to (e_1, e_2). Obviously, $\mathcal{L}^{(0,0)} \cong \mathcal{W}$.

[2] Here $\bar{z}$ does not denote conjugation, but taking the residue class modulo the lattice.

Proposition 5.1 ([6, *Proposition* 5.1]) *For* $(e_1, e_2) \neq (0,0)$ *the algebras* $\mathcal{L}^{(e_1,e_2)}$ *are not isomorphic to the Witt algebra* $\mathcal{W}$, *but* $\mathcal{L}^{(0,0)} \cong \mathcal{W}$.

If we restrict our two-dimensional family to a line D_s ($s \neq \infty$) then we obtain a one-dimension family

$$[V_n, V_m] = \begin{cases} (m-n)V_{n+m}, & n, m \text{ odd}, \\ (m-n)\big(V_{n+m} + 3e_1 V_{n+m-2} \\ \quad + e_1^2(1-s)(2+s)V_{n+m-4}\big), & n, m \text{ even}, \\ (m-n)V_{n+m} + (m-n-1)3e_1 V_{n+m-2} \\ \quad + (m-n-2)e_1^2(1-s)(2+s)V_{n+m-4}, & n \text{ odd}, \ m \text{ even}. \end{cases} \tag{5.13}$$

Here s has a fixed value and e_1 is the deformation parameter. (A similar family exists for $s = \infty$.) It can be shown that as long as $e_1 \neq 0$ the algebras over two points in D_s are pairwise isomorphic but not isomorphic to the algebra over 0, which is the Witt algebra. Using the result $\mathrm{H}^2(\mathcal{W}, \mathcal{W}) = \{0\}$ of Fialowski [4] we get

Theorem 5.2 *Despite its infinitesimal and formal rigidity the Witt algebra* $\mathcal{W}$ *admits deformations* $\mathcal{L}_t$ *over the affine line with* $\mathcal{L}_0 \cong \mathcal{W}$ *which restricted to every* (*Zariski or analytic*) *neighborhood of* $t = 0$ *are non-trivial.*

The one-dimensional families (5.13) are examples of jump deformations as $\mathcal{L}_t \cong \mathcal{L}_{t'}$ for $t, t' \neq 0$. The isomorphism is given by rescaling the basis elements. This is possible as long as $e_1 \neq 0$. In fact, using $V_n^* = (\sqrt{e_1})^{-n} V_n$ (for $s \neq \infty$) we obtain for e_1 always the algebra with $e_1 = 1$ in the structure equations (5.13).

Using the cocycle (4.10) in the families (5.12), (5.13) a central term can be easily incorporated. With respect to the flat coordinate $z - a$ we can take the projective connection $R \equiv 0$. The integral along a separating cocycle C_S is obtained by taking the residue at $z = 0$. In this way we obtain geometric families of deformations for the Virasoro algebra. They are locally non-trivial despite the fact that the Virasoro algebra is formally rigid.

5.3 A Family Which is Not a Jump Deformation

Let us stress the fact that the two-dimensional family (5.12) is not a jump deformation. In fact there exist even one-dimensional deformations as subfamilies which are not jump deformations. Take for example the smooth rational curve given by

$$C := \{(e_1, e_2) \in \mathbb{C} \mid e_2 = 2e_1^2, \ e_1 \in \mathbb{C}\}. \tag{5.14}$$

The rational parameter will be e_1. Automatically we have $e_3 = -(1 + 2e_1)e_1$.

The curve passes through $(0,0)$. For every line D_s there will be just one other point of intersection with C. Its parameter value is given by $e_1 = 1/2s$. Hence for e_1 small the curve will not meet the exceptional lines D_s, $s = 1, -1/2, -2$ a second time. The curves corresponding to small $e_1 \neq 0$ values will be nonsingular cubics, i.e. elliptic curves.

The elliptic modular function classifying elliptic curves up to isomorphisms is given by

$$j = 1728 \frac{g_2^3}{\Delta}. \tag{5.15}$$

Expressing the j-function in terms of e_i, $i = 1, 2, 3$ and substituting $e_3 = -(e_1 + e_2)$ yields

$$j(e_1, e_2) = 1728 \, \frac{4(e_1^2 + e_1 e_2 + e_2^2)^3}{(e_1 - e_2)^2 (2e_1 + e_2)^2 (e_1 + 2e_2)^2}. \tag{5.16}$$

First we evaluate this along D_s, $s \neq 1, -1/2, -2$ and obtain

$$j(s, e_1) = 1728 \frac{4(1 + s + s^2)^3}{(1 - s)^2 (2 + s)^2 (1 + 2s)^2}, \qquad j(\infty) = 1728. \tag{5.17}$$

As this does not depend on the parameter e_1 anymore it follows that the elliptic curves over $D_s \setminus \{0\}$ for a fixed s are isomorphic, in accordance with the fact that these deformations yield jump deformations. A remark aside: the poles of $j(s)$ correspond exactly to the exceptional lines. They correspond to nodal cubics, see Sect. 6.

Next we evaluate j along the curve C and obtain

$$j(e_1) = 1728 \frac{(1 + 2e_1 + 4e_1^2)^3}{(1 - 2e_1)^2 (1 + e_1)^2 (1 + 4e_1)^2}. \tag{5.18}$$

This value will not be constant along C. Furthermore, for small e_1 the values will be different. This implies that the elliptic curves will be pairwise non-isomorphic. The vector field algebras along this curve in the neighborhood of 0 will also be pairwise non-isomorphic.

By specializing $e_2 = 2e_1^2$ and $e_3 = -e_1(1 + 2e_1)$ in (5.12) we obtain for the vector field algebra

$$[V_n, V_m] = \begin{cases} (m-n) V_{n+m}, & n, m \text{ odd}, \\ (m-n)\big(V_{n+m} + 3e_1 V_{n+m-2} \\ \quad + 2e_1(1-2e_1)(1+e_1) V_{n+m-4}\big), & n, m \text{ even}, \\ (m-n) V_{n+m} + (m-n-1) 3e_1 V_{n+m-2} \\ \quad + (m-n-2) 2e_1 (1-2e_1)(1+e_1) V_{n+m-4}, & n \text{ odd}, \ m \text{ even}. \end{cases} \tag{5.19}$$

5.4 The Current Algebra

Let $\mathfrak{g}$ be a simple finite-dimensional Lie algebra (similar results are true for general Lie algebras) and $\mathcal{A}$ the algebra of meromorphic functions corresponding to the geometric situation discussed above. A basis for $\mathcal{A}$ is given by

$$A_{2k} = (X - e_1)^k, \qquad A_{2k+1} = \frac{1}{2} Y \cdot (X - e_1)^{k-1}, \quad k \in \mathbb{Z}. \tag{5.20}$$

We calculate for the elements of $\overline{\mathcal{G}}$

$$[x \otimes A_n, y \otimes A_m] = \begin{cases} [x, y] \otimes A_{n+m}, & n \text{ or } m \text{ even}, \\ [x, y] \otimes A_{n+m} + 3e_1 [x, y] \otimes A_{n+m-2} \\ \ +(e_1 - e_2)(2e_1 + e_2)[x, y] \otimes A_{n+m-4}, & n \text{ and } m \text{ odd}. \end{cases} \tag{5.21}$$

If we let e_1 and e_2 (and hence also e_3) go to zero we obtain the classical current algebra as degeneration. Again it can be shown that the family, even if restricted on D_s, is locally non-trivial, see [7]. Recall that by results of Lecomte and Roger [14] the current algebra is formally rigid if $\mathfrak{g}$ is simple. But our families show that it is neither geometrically nor analytically rigid.

The families over D_s are jump deformations. But again there exists one-dimensional subfamilies of deformations of (5.21) which are non-trivial and not jump deformations.

Also in this case we can construct families of centrally extended algebras by considering the cocycle (4.11). In this way we obtain non-trivial deformation families for the formally rigid classical affine algebras of Kac–Moody type. The cocycle (4.11) is

$$\gamma(x \otimes A_n, y \otimes A_m) = p(e_1, e_2) \cdot \beta(x, y) \cdot \frac{1}{2\pi \mathrm{i}} \int_{C_S} A_n dA_m. \tag{5.22}$$

Here $p(e_1, e_2)$ is an arbitrary polynomial in the variables e_1 and e_2. and β the Cartan–Killing form. The integral can be calculated [7, Theorem 4.6] as

$$\frac{1}{2\pi \mathrm{i}} \int_{C_S} A_n dA_m = \begin{cases} -n\delta_m^{-n}, & n, m \text{ even}, \\ 0, & n, m \text{ different parity}, \\ -n\delta_m^{-n} + 3e_1(-n+1)\delta_m^{-n+2} & \\ \quad + (e_1 - e_2)(2e_1 + e_2)(-n+2)\delta_m^{-n+4}, & n, m \text{ odd}. \end{cases} \tag{5.23}$$

6 The Geometric Interpretation

If we take $e_1 = e_2 = e_3$ in the definition of the cubic curve (5.8) we obtain the cuspidal cubic E_C with affine part given by the polynomial $Y^2 = 4X^3$. It has a singularity at $(0, 0)$ and the desingularization is given by the projective line $\mathbb{P}^1(\mathbb{C})$. This says there exists a surjective (algebraic) map $\pi_C : \mathbb{P}^1(\mathbb{C}) \to E_C$ which outside the singular point is $1:1$. Over the cusp lies exactly one point. The vector fields, resp. the functions, resp. the $\mathfrak{g}$-valued functions can be degenerated to E_C and pull-backed to vector fields, resp. functions, resp. $\mathfrak{g}$-valued functions on $\mathbb{P}^1(\mathbb{C})$. The point $(e_1, 0)$ where a pole is allowed moves to the cusp. The other point stays at infinity. In particular by pulling back the degenerated vector field algebra we obtain the algebra of vector fields with two possible poles, which is the Witt algebra. And by pulling back the degenerated current algebra we obtain the classical current algebra.

The exceptional lines D_s for $s = 1, -1/2, -2$ are related to interesting geometric situations. Above $D_s \setminus \{(0, 0)\}$ with these values of s, two of the e_i are the same, the third one remains distinct. The curve will be a nodal cubic E_N defined by $Y^2 = 4(X - e)^2(X - e)$. The singularity will be a node with the coordinates $(e, 0)$. Again the desingularization will be the projective line $\pi_N : \mathbb{P}^1(\mathbb{C}) \to E_N$. But now above the node there will be two points in $\mathbb{P}^1(\mathbb{C})$. For the pull-backs we have the following two situations:

(1) If $s = 1$ or $s = -2$ then $e = e_1$ and the node is a possible point for a pole. We obtain objects on $\mathbb{P}^1(\mathbb{C})$ which beside the pole at ∞ might have poles at two other places. Hence, we obtain a three-point Krichever–Novikov algebra of genus 0.
(2) If $s = -1/2$ then at the node there is no pole. The number of possible poles for the pull-back remains two. We obtain certain subalgebras of the classical two point case. Additionally, for the vector field case we have to pay attention to the fact that the vector fields obtained by pull-back acquire zeros at the points lying above the node.

These algebras were identified and studied in detail in [6, 7, 22].

The deformed families are of importance for the quantization of conformal field theories. In this context the behavior of objects when we approach the boundary of the moduli space of curves with marked points has to be studied.

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Appendix 1 Definition of a Deformation

In the following we will assume that A is a commutative algebra over $\mathbb{K}$ (where $\mathbb{K}$ is a field of characteristic zero) which admits an augmentation $\epsilon : A \to \mathbb{K}$. This says that ϵ is a $\mathbb{K}$-algebra homomorphism, e.g. $\epsilon(1_A) = 1$. The ideal $m_\epsilon := \operatorname{Ker} \epsilon$ is a maximal ideal of A. Vice versa, given a maximal ideal m of A with $A/m \cong \mathbb{K}$, the natural quotient map defines an augmentation.

If A is a finitely generated $\mathbb{K}$-algebra over an algebraically closed field $\mathbb{K}$ then $A/m \cong \mathbb{K}$ is true for every maximal ideal m. Hence, in this case every such A admits at least one augmentation and all maximal ideals are coming from augmentations.

Let us consider a Lie algebra $\mathcal{L}$ over the field $\mathbb{K}$, ϵ a fixed augmentation of A, and $m = \operatorname{Ker} \epsilon$ the associated maximal ideal.

Definition 7.1 ([5]) A *deformation* λ of $\mathcal{L}$ with base (A, m) or simply with base A, is a Lie A-algebra structure on the tensor product $A \otimes_{\mathbb{K}} \mathcal{L}$ with bracket $[.,.]_\lambda$ such that

$$\epsilon \otimes \mathrm{id} : A \otimes \mathcal{L} \to \mathbb{K} \otimes \mathcal{L} = \mathcal{L} \tag{7.1}$$

is a Lie algebra homomorphism.

Specifically, it means that for all $a, b \in A$ and $x, y \in \mathcal{L}$,

(1) $[a \otimes x, b \otimes y]_\lambda = (ab \otimes \mathrm{id})[1 \otimes x, 1 \otimes y]_\lambda$,
(2) $[.,.]_\lambda$ is skew-symmetric and satisfies the Jacobi identity,
(3) $\epsilon \otimes \mathrm{id}([1 \otimes x, 1 \otimes y]_\lambda) = 1 \otimes [x, y]$.

By Condition (1) to describe a deformation it is enough to give the elements $[1 \otimes x, 1 \otimes y]_\lambda$ for all $x, y \in \mathcal{L}$. If $B = \{z_i\}_{i \in J}$ is a basis of $\mathcal{L}$ it follows from Condition (3) that the Lie product has the form

$$[1 \otimes x, 1 \otimes y]_\lambda = 1 \otimes [x, y] + \sum_i{}' a_i \otimes z_i, \tag{7.2}$$

with $a_i = a_i(x, y) \in m$, $z_i \in B$. Here $\sum'$ denotes a finite sum. Clearly, Condition (2) is an additional condition which has to be satisfied.

If we use $A = \mathbb{C}[t]$ we get exactly the notion of a one parameter geometric deformation discussed above.

A deformation is called *trivial* if $A \otimes_{\mathbb{K}} \mathcal{L}$ carries the trivially extended Lie structure, i.e. (7.2) reads as $[1 \otimes x, 1 \otimes y]_\lambda = 1 \otimes [x, y]$.

Two deformations of a Lie algebra $\mathcal{L}$ with the same base A are called *equivalent* if there exists a Lie algebra isomorphism between the two copies of $A \otimes \mathcal{L}$ with the two Lie algebra structures, compatible with $\epsilon \otimes \mathrm{id}$.

Formal deformations are defined in a similar way. Let A be a complete local algebra over $\mathbb{K}$, so $A = \overleftarrow{\lim}_{n\to\infty}(A/m^n)$, where m is the maximal ideal of A. Furthermore, we assume that $A/m \cong \mathbb{K}$, and $\dim(m^k/m^{k+1}) < \infty$ for all k.

Definition 7.2 ([3]) A *formal deformation* of $\mathcal{L}$ with base A is a Lie algebra structure on the completed tensor product $A \widehat{\otimes} \mathcal{L} = \overleftarrow{\lim}_{n\to\infty}((A/m^n) \otimes \mathcal{L})$ such that

$$\epsilon \widehat{\otimes} \mathrm{id} : A \widehat{\otimes} \mathcal{L} \to \mathbb{K} \otimes \mathcal{L} = \mathcal{L} \tag{7.3}$$

is a Lie algebra homomorphism.

If $A = \mathbb{C}[[t]]$, then a formal deformation of $\mathcal{L}$ with base A is the same as a formal one parameter deformation discussed above. There is an analogous definition for equivalence of deformations parameterized by a complete local algebra.

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